Leibnitz Theorem.

Date ______
Page ______

Leibnitz Theorem:- If u and v are the function of x ~~then~~, such that $y = uv$. Then, nth derivative of y i.e $y_n = \frac{d^n y}{dx^n}$ is given by

$$y_n = \frac{d^n y}{dx^n} = {}^nC_0\, u_n \cdot v + {}^nC_1\, u_{n-1} \cdot v_1 + {}^nC_2\, u_{n-2} \cdot v_2 + {}^nC_3\, u_{n-3} \cdot v_3 + \cdots {}^nC_r\, u_{n-r} v_r \cdots + {}^nC_n \cdot u \cdot v_n.$$

Proof: Let $y = u \cdot v$.

where
$u = f(x)$
$v = f(x)$
${}^nC_r = \frac{n!}{r!(n-r)!}$

Now,
Formula Binomial expansion!

$$(a+b)^n = {}^nC_0\, a^n b^0 + {}^nC_1\, a^{n-1} b^1 + {}^nC_2\, a^{n-2} b^2 + {}^nC_3\, a^{n-3} b^3 + \cdots {}^nC_r\, a^{n-r} \cdot b^r + \cdots {}^nC_n\, a^{n-n} \cdot b^n$$

$$(a+b)^n = {}^nC_0\, a^n b^0 + {}^nC_1 \cdot a^{n-1} \cdot b^1 + {}^nC_2 \cdot a^{n-2} b^2 + {}^nC_3 \cdot a^{n-3} b^3 + {}^nC_r \cdot a^{n-r} \cdot b^r + \cdots {}^nC_n\, a b^n.$$

Second method:

Let $y = u \cdot v$ where u and v are the function of x.

On differentiating w.r.to x we get:

Date
Page

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

also, $Dy = uDv + vDu$ where $D = \frac{d}{dx}$ called differential operator.

$Dy = u.Dv + vDu$. ——— (I)

Let $Du = D_1 u$

and $Dv = D_2 v$.

Then equation (I) take the form,

$Dy = uD_2 v + vD_1 u$

$= D_2 uv + D_1 uv$. ——— (II)

$Dy = (D_1 + D_2)\, u.v$.

∵ we know that

$Dy = y_1$

$D^2 y = y_2$

$D^3 y = y_3$

$D^4 y = y_4$

⋮

$D^n y = y_n$.

So, on taking the nth derivative of (II) we gets

$D^n y = (D_1 + D_2)^n u.v$

Now, using Binomial theorem we gets

$D^n y = y_n = {}^nC_0 D_1^n D_2^0 uv + {}^nC_1 D_1^{n-1} D_2^1 uv + {}^nC_2 D_1^{n-2} D_2^2 uv + \cdots {}^nC_r D_1^{n-r} D_2^r \cdots {}^nC_n D_1^0 D_2^n uv.$

$= {}^nC_0 D_1^n uv + {}^nC_1 D_1^{n-1} D_2 uv + {}^nC_2 D_1^{n-2} D_2^2 + \cdots$

$\cdots {}^nC_r D_1^{n-r} D^r uv + {}^nC_n D_2 uv$

$\therefore\ {}^nC_0 = 1$

Date ______
Page ______

$$= V \cdot D_1^n u + {}^nC_1 D_1^{n-1} u\, D_2^1 V + {}^nC_2 D_1^{n-2} u \cdot D_2^2 V +$$

$$\ldots {}^nC_r D_1^{n-r} u\, D_2^r V + \ldots {}^nC_n u\, D_2^n V$$

$\therefore$ We know that

$D_1^n u = u_n \longrightarrow$ nth time derivative

Now, also, $D_1^{n-1} u = u_{n-1}$

$$\therefore D^n y = u_n \cdot V + {}^nC_1 u_{n-1} V_1 + {}^nC_2 u_{n-2} V_2$$

$$+ {}^nC_r u_{n-r} \cdot V_r + {}^nC_n \cdot V_n$$

Proved.

Q. WORKED OUT ~~[illegible]~~ QUESTION

Q. If $y = x^2 \cdot e^{ax}$, find y_n

Solution:- Let $u = e^{ax}$

$V = x^2$

$\therefore V_1 = 2x,\ V_2 = 2,\ V_3 = 0,\ V_4 = 0$

$u_1 = e^{ax} \cdot a \quad u_2 = e^{ax} a^2,\ u_3 = a^3 e^{ax} \ldots$

$\therefore u_n = a^n e^{ax}$

$$\therefore y_n = {}^nC_0 \cdot u_n \cdot V + {}^nC_1 u_{n-1} V_1 + {}^nC_2 \cdot u_{n-2} \cdot V_2 + \ldots {}^nC_n V_n.$$

On putting the value of u_n, V, V_2 for above equation we get

$$y_n = 1 \cdot a^n e^{ax} x^2 + n \cdot a^{n-1} e^{ax} 2x + \frac{n(n-1)}{2} \cdot a^{n-2} e^{ax} \cdot 2 + 0 + 0$$

$$y_n = a^n \cdot e^{ax} x^2 + 2n a^{n-1} x e^{ax} + \frac{\cancel{2}\, n(n-1)}{\cancel{2}} a^{n-2} e^{ax}$$

$$= a^n e^{ax} x^2 + 2n a^{n-1} x \cdot e^{ax} + n(n-1) \cdot a^{n-2} \cdot e^{ax}$$

Ans.